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Walks on Apollonian networks

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Abstract. We carry out comparative studies of random walks on deterministic Apollonian networks (DANs) and random Apollonian networks (RANs). We perform computer simulations for the mean first-passage time, the average return time, the mean-square displacement, and the network coverage for the unrestricted random walk. The diffusions both on DANs and RANs are proved to be sublinear. The effects of the network structure on the dynamics and the search efficiencies of walks with various strategies are also discussed. Contrary to intuition, it is shown that the self-avoiding random walk, which has been verified as an optimal local search strategy in networks, is not the best strategy for the DANs in the large size limit.

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In the past few years, much attention has been devoted to the characterization and modelling of a wide range of complex systems that can be described as networks [1–3]. The topological properties of real-world networks have been studied extensively. But an even more intriguing task, and a natural extension of these studies, is to understand how the topological structure of networks affects dynamics taking place on top of them [4]. Many dynamical processes have been studied on complex networks, such as epidemic spreading [5], percolation [6], synchronization [7], and so on. These works have shown that topologies of networks play an important role in determining the system dynamical features.

Random walk has been used for modelling various dynamics in physical, biological, and social contexts [8]. It could also be a mechanism of transport and search on networks [9–11] when no knowledge of the global properties of the underlying networks is available. Thus one interesting problem is to study the dynamical behavior of a random walker on networks with different topological properties. Much is known about random walks on both regular and random networks [12,13]. In addition, there have been several recent studies of random walks on small-world networks (SWNs) [14–17] and scale-free networks (SFNs) [9,18–20]. The impacts of the heterogeneous topological structures of the networks on the nature of the diffusive and relaxation dynamics of the random walk have also been probed recently [9,10,21,22].

In this paper we investigate walk processes taking place upon the deterministic Apollonian network (DAN) [23] and its variation, the random Apollonian network (RAN) [24]. The DAN can be defined based on the ancient problem of filling space with spheres, first tackled by the Greek mathematician Apollonius of Perga [25]. That is, starting with an initial array of touching disks, which have curvilinear-triangle interstices, disks are added inside each existing interstice in the present configuration, such that these disks touch each of the disks bounding the curvilinear triangles. Each of these added disks give rise to three smaller interstices, which will be filled in the next generation. This process is then repeated for successive generations. The DAN is constructed based on this process by considering each disk as a node, and the disks in contact as the corresponding nodes connected. For each new node added to a certain triangle (corresponding to the curvilinear-triangle interstice) and linked to the three vertices, three new triangles are created in the network, into which nodes will be inserted in the next generation. The DAN is a typical regular network, which has deterministic size when the number of its generations is certain. Different from recursive constructing of the DAN [23], the RAN starts with a triangle containing three nodes. Then, at each time step, only one triangle is randomly selected to add a new node linking to its three vertices [24]. Both networks are simultaneously scale-free, small-world, Euclidean, and space filling. They have attracted increasing interest recently [26,27].

We carry out the walk along the bonds of a given network as follows: (i) there is only one walker on the network at a time; (ii) the random walker is injected onto a randomly chosen node on the network, a new node for

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each walker. We will call this node the "origin" of the walk; (iii) at each discrete time step t, the walker will hop to one nearest neighbor of its current node according to one certain strategy. The walk strategies adopted by the walker include the following: unrestricted random walk (RW), no-back (NB) walk, no-triangle-loop (NTL) walk, no-quadrangle-loop (NQL) walk, and self-avoiding (SA) random walk. For the RW, the walker may unrestrictedly hop to a nearest-neighbor node by randomly taking one of the links. It forgets all information about its past. The NB walk implies that a random walker, if possible, will not return to the node it was situated at the previous step. Similarly, the NTL and NQL random walks mean that the walker will try to avoid walking in loops, with three or four edges, respectively, unless there is no other choice. We note that the NQL walk also includes the NTL, which means it eliminates quadrangle loops as well as triangle loops. Finally, the definition of the SA walk as a search strategy in this paper is different from that of the former SA walk and the kinetic growth self-avoiding (KGSA) walk [28–31], both of which were used in polymer statistics. In the former SA walk one randomly chooses the next step from among the nearest-neighbor nodes (excluding the previous one); if it happens that one chooses an already visited node, the walk stops. In the KGSA walk [29] one instead randomly chooses the next step from among the nearest-neighbor *unvisited* nodes and stops growing only when none are available. Here, defined as a searcher, the SA walker tries to avoid revisiting nodes in the same way as the KGSA walker, however it stops on finding the target, allowing *compulsive revisit* if no nearest-neighbor unvisited node is left [20] (hops to a nearest-neighbor node by randomly taking one of the links). Clearly, this SA walk also includes the NTL and NQL walks.

In the following, we will investigate walk processes on the DANs and RANs. The corresponding results on the Watts-Strogatz (WS) ($K = 3$, $p_0 = 0.1$) [32] and Barabási-Albert (BA) ($m = m_0 = 3$) [33] networks are also presented for comparison. Note that the four networks have the same average degree $\langle k\rangle = 6$.

In the context of transport and search, which intimately relate to random walk mechanism, the mean first-passage time (MFPT) and its special case, the average return time (ART) are important characteristics. The MFPT of a random walker from origin i to another node j is denoted by $\langle T_{ij}\rangle$. The ART, i.e., the average time a walker needs to return to the origin i, is denoted by $\langle T_{ii}\rangle$. For a finite network which consists of N nodes, with i and j belong to $[1, N]$, Noh and Rieger [18] have proved that the MFPT is negatively correlated with K_j (the degree of node j), and $\langle T_{ii}\rangle$ has a very simple form as

$$\langle T_{ii}\rangle = \mathcal{N}/K_i \tag{1}$$

with $\mathcal{N} = \sum_j K_j$ ($j = 1,\dots,N$) and K_i the degree of node i. That is to say, nodes with higher degrees are visited earlier and more frequently, and then targets on these nodes can be found more easily than on nodes with smaller degrees. We simulate the MFPT and ART of each node for the DAN and RAN with $N = 9844$ nodes (9 generations of the DAN). The MFPT of a given node j denoted by $\langle T_j\rangle$ is defined as,

$$\langle T_j\rangle = \frac{\sum_{i=1,i\neq j}^{N}\langle T_{ij}\rangle}{N-1} \tag{2}$$

with all other nodes i in the network as the origin, so that the choice of node i will not influence the result of $\langle T_j\rangle$. After averaging over the MFPT and ART of all the nodes for a given degree k, the results as a function of node degree are depicted in Figure 1. We can see that, the linear property of the log-log plot of ART (the lower panels) is in excellent agreement with equation (1), and the negative correlation between MFPT and k is also present (the upper panels).

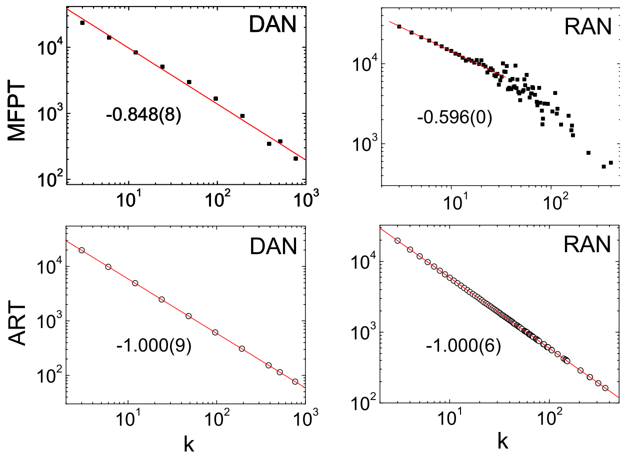

Fig. 1. (Color online) The mean first-passage times $\langle T_{ij}\rangle$ (upper panels) and the average return time $\langle T_{ii}\rangle$ (lower panels) averaged over the results of given k's, for the DANs (left) and RANs (right). Fitted linear relations of average return time (solid lines) are obtained with a slope of -1 for both DANs and RANs.

Then, let us discuss the search efficiencies of the above-mentioned five walk strategies on the four networks. Designing efficient search strategies in networks is an important issue related to random walks [9,18,20]. Here we suppose that, at every step, the walker adopting various strategies can inspect the nearest neighbors of its present node; if the target is at one of them, this round of search is over. The search time is defined as the average number of steps needed to complete the search. The search processes with the five strategies are performed on networks with different sizes, and the search times are averaged over 50 realizations of the networks and 200 randomly selected couples of origins and targets for each of them. One can find some common features in Figure 2. For example, it is remarkable that the search times of various strategies on the four networks have the same scaling properties with the slope approximately equal to 1 (except for the SA walk in the DANs). As another example, the RW walk is always the most inefficient search strategy for all the four networks.

There still exist some obvious differences among them. We observe that the DANs are somewhat well suited to

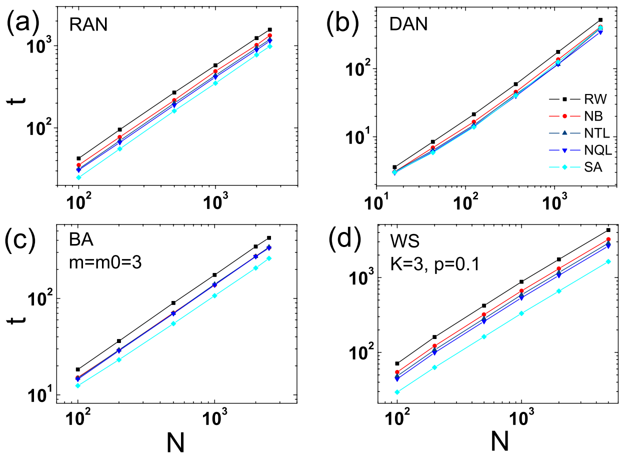

Fig. 2. (Color online) Average search times of various search strategies on the RANs (a), DANs (b), BA networks (c), and WS networks (d) with average degree $\langle k \rangle = 6$. The slopes of all the scaling relationships are approximately equal to 1.

search, with the average search times of all the five strategies smaller than those on the other three networks. On the other hand, only for the BA network, the search efficiencies of the NB, NTL and NQL walks (see Fig. 2c, and also Ref. [20]) nearly collapse into one. One can imagine that an unrestricted RW walker may be trapped into a local region and revisit node there by reasons of reversal and walking in loops. However the searchers, such as the NB, NTL and NQL walkers who avoid revisiting nodes, can escape from those regions more easily, and thus improve the search efficiencies. Furthermore, in the networks with weak clustering effect, where loops do not prevail, the NTL and NQL walkers, who adopt increasingly stricter rules to avoid revisiting nodes than the NB walker, can not capitalize on their advantages. Therefore, their search efficiencies keep very close to or even collapse into that of the NB walker. As a consequence, we can conclude that different clustering properties of the underlying networks induce efficiency differences among the above mentioned strategies. In return, the efficiencies of those search strategies will reflect the clustering effect of the networks. The collapse on the BA network is due to its smallest clustering coefficient in the four networks. For the RANs and DANs (see Figs. 2a and 2b), the efficiency improvements of the NTL and NQL walks compared to the NB walk reflect the fact that high clustered nodes are popular in them.

The SA walk was proved to be generally the most efficient strategy if the walker is not aware of the global structure of the underlying network [20]. However, a counterexample is the SA walk on the DANs. Figure 2b shows the average search times on the DANs with generations ranging from 3 to 8, corresponding to network sizes 16, 43, 124, 367, 1096, and 3283, respectively. In contrast to the conclusions obtained in reference [20], the SA walk does not markedly reduce the search time and performs even more inefficiently than the NTL and NQL walks when there are more than 6 generations of the DANs.

Table 1. The critical threshold P_c for PSP and average path length $\langle L \rangle$ of the four networks with $N = 9844$ nodes.

	DAN	RAN	BA	WS
$P_c \approx$	$4/N$	0.03	0.3	0.63
$\langle L \rangle$	4.06(5)	5.40(9)	4.27(9)	8.61(9)

We argue that the following factor should be taken into account to understand the bad performance of the SA walk on the DANs. According to the former analysis of MFPT, we knew that the nodes with higher degrees would be visited earlier by the walker, and thus would have a larger probability to be avoided earlier if the walker adopts the SA strategy. In other words, the SA walker may preferentially avoid high-degree nodes, which is similar to the process of the intentional removals aiming at these nodes. Therefore, to some extent, the SA search process on networks can be considered as an example of initentional attack [34] or preferential site percolation (PSP) on networks. The intentional attack means the removal of nodes and their incident edges targeting on nodes with high degrees, and the critical threshold for PSP denoted by P_c is a measure of how stable the network is against this attack. When the fraction P of the removed nodes exceeds P_c, the network disintegrates into smaller, disconnected fragments. The DANs and RANs are much frailer than the BA and WS networks under intentional attack (see Tab. 1 for P_c of networks with $N = 9844$ as an example), because of the crucial importance of the high-degree nodes to the network integrity. Moreover, when the network size is sufficiently large (e.g. composing of more than 6 generations' nodes), the DAN is more sensitive to intentional attack than the RAN [24].

We know that the walker, if rigorously avoiding visiting those crucial high-degree nodes, can never reach some regions of the network from its current position, as if the network is intentionally attacked and disintegrates into parts, e.g. the walker avoiding visiting the first 4 greatest-degree nodes of the DAN will wander only in one of the three disconnected clusters. Similarly, for the SA walker, the search is indeed slowed down by avoiding revisiting those nodes. A direct mapping from the search process of the SA walk to the question of the intentional attack is not rigorous, due to the compulsive revisiting which ensures the accessibility to the targets. However, considering the similarity between them, we can still conclude qualitatively that the SA search strategy, compared to other mentioned strategies, will improve the search efficiency to certain extent for the networks with higher critical threshold.

Thus, when designing efficient search strategies, one should not merely try to intensify the elimination of revisiting, but consider the factual topology of the underlying networks, such as the extent of clustering effect and the role of the high-degree nodes to the network integrity.

Next, we will present our simulation results for two quantities, the mean-square displacement and the network coverage. The mean-square displacement $\langle R^2(t) \rangle$ of a particle diffusing in a given space, which is a measure of the

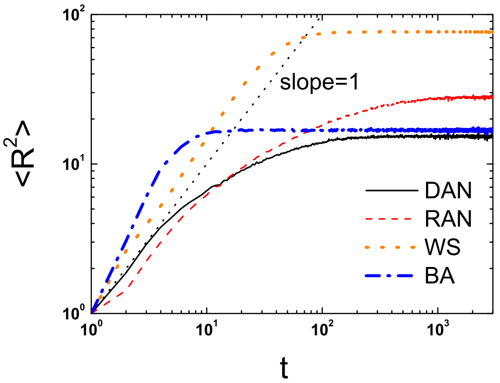

Fig. 3. Mean-square distance $\langle R^2 \rangle$ as a function of time for the DANs, RANs, WS and BA networks with $N = 9844$ and average degree $\langle k \rangle = 6$ fixed. The dotted line with slope 1 is plotted for comparison.

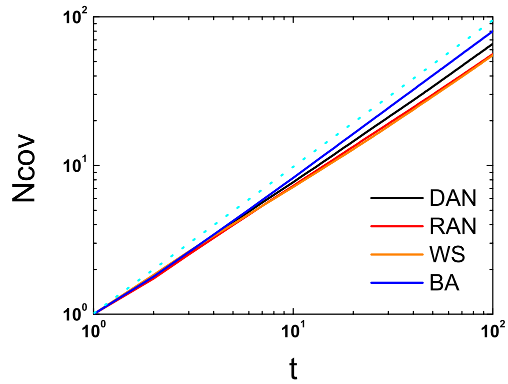

Fig. 4. (Color online) Network coverage N_{cov} after t steps on (solid lines, top to bottom) BA networks, DANs, RANs and WS networks with $N = 9844$ and $\langle k \rangle = 6$ fixed.

distance R covered by an unrestricted RW walker after performing t steps, is one of the most basic quantities in random walk theory [16,19]. In most cases, this quantity is described by an expression of the form

$$\langle R^2(t) \rangle \sim t^a. \qquad (3)$$

The value of the parameter a classifies the type of diffusion into normal linear diffusion ($a = 1$), subdiffusion ($a > 1$), or superlinear diffusion ($a < 1$). When we consider distinct time steps and nearest-neighbor hops on networks, the maximum allowable value of a is 2 [19]. Recently, mean squared displacement was studied in SWNs and SFNs [16,19]. It was shown that diffusion on their small-world network model [16] is linear, and diffusion on SFNs, by varying the value of the degree exponent γ, may range from superlinear to sublinear [19].

To calculate $\langle R^2(t) \rangle$ we first, at each time step, find the minimal distance from the current position of the walker to the origin (i.e., the smallest number of steps needed for the walker to reach the origin) using a breadth-first search method. Then we allow the walker to move through the network until $\langle R^2(t) \rangle$ has saturated. Finally, the results are averaged over different origins of the walkers and realizations of the network. We simulate $\langle R^2(t) \rangle$ for the DAN, RAN, WS and BA networks with $N = 9844$ nodes, and report the results as a function of MC time in Figure 3. One important feature is the fact that $\langle R^2(t) \rangle$ finally equilibrates to a constant displacement value. This is a simple manifestation of the small diameter of these finite networks. Note also that, because of the differences of the average path length $\langle L \rangle$ of these networks (see Tab. 1 for $\langle L \rangle$), the plateau values are also different. For the DANs and RANs, one can find that the slopes of $\langle R^2 \rangle$ are comparatively small, especially that of RANs. This can be explained as a result of their high clustering effect which induces the walker to spend much time exploring the clusters in the networks, and thus the distance to the origin increases slowly. From the slope of $\langle R^2 \rangle$, we know that diffusion both on the DANs and RANs are sublinear (the value of the slope is smaller then 1), and diffusion both on the BA and WS networks are superlinear (the value of the slope is larger then 1). The network coverage denoted by N_{cov} is defined as the average number of distinct visited nodes of the RW walker. Results of N_{cov} on the four networks are presented in Figure 4. In order to decrease finite size effects, the number of steps performed is nearly two orders of magnitude smaller than the size of the networks. The clustering effects of DANs and RANs are exhibited by their relatively low coverage compared to BA networks.

In summary, we present comparative studies of the dynamics of random walks on DANs and RANs. From the search efficiencies of various strategies simulated in this paper, we find that the clustering effect of networks can result in efficiency differences between the NB, NTL and NQL walks, and likewise, these differences in return can reflect the clustering properties of networks. For DANs with large size, the SA random walk is no longer the best search strategy, which is shown to be due to the crucial importance of the high-degree nodes to the network integrity. Thus, while optimizing the search strategy in networks, one should take into account the topological properties of the underlying network, including clustering and the significance of the high-degree nodes. Since search is a problem of extreme importance for so many natural and artificial networks, this finding may be of practical value. Finally, the simulation results of mean-square displacement $\langle R^2(t) \rangle$ and network coverage N_{cov} also show the influence of the structure of networks on the dynamics of random walks.

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